

q-deformed Fermions

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Abstract

This is a study of q -Fermions arising from a q -deformed algebra of harmonic oscillators. Two distinct algebras will be investigated. Employing the first algebra, the Fock states are constructed for the generalized Fermions obeying Pauli exclusion principle. The distribution function and other thermodynamic properties such as the internal energy and entropy are derived. Another generalization of fermions from a different q -deformed algebra is investigated which deals with q -fermions not obeying the exclusion principle. Fock states are constructed for this system. The basic numbers appropriate for this system are determined as a direct consequence of the algebra. We also establish the Jackson Derivative, which is required for the q -calculus needed to describe these generalized Fermions.

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I. INTRODUCTION

We shall investigate q -deformed Fermions arising as a consequence of the q -deformed algebra of harmonic oscillators. We shall study two distinct algebras.

First we shall consider generalized Fermions obeying the algebra $a^\dagger + q^{-1}a^\dagger a = q^{-N}$, $0 \leq q \leq 1$. These generalized Fermions will be shown to obey the exclusion principle, with the Fock states restricted to $n = 0, 1$ only. We shall also see that this algebra is not associated with basic numbers. We shall investigate in detail, the statistical thermodynamics of these Fermions which require the use of ordinary derivatives rather than the Jackson Derivative (JD) of q -calculus. It will be shown that despite the fact that they obey the exclusion principle, the thermodynamic properties are quite different from that of ordinary Fermions.

We shall also investigate q -deformed Fermions arising from the oscillator algebra $aa^\dagger + qa^\dagger a = q^{-N}$, $0 \leq q \leq 1$. It will be shown that these generalized Fermions do not obey the exclusion principle and the Fock states consist of $n = 0, 1, 2, 3, \dots$ with arbitrary number of quanta. We shall not investigate the thermodynamics of these Fermions, and we shall confine ourselves to a study of the Fock states and some general properties. We shall establish the JD needed for the q -calculus governing this system.

Although this investigation, together with a corresponding q -deformed Bosons, accommodates an interpretation as an interpolating statistics such as in [1], we shall strictly regard this as a study of generalized Fermions. The first algebra investigated here is similar to what is discussed in an earlier work [2] but there are significant differences as we shall show.

II. Q -FERMIONS OBEYING EXCLUSION PRINCIPLE

Let us begin with the algebra defined by

$$aa^\dagger + q^{-1}a^\dagger a = q^{-N}, \quad 0 \leq q \leq 1, \quad (1)$$

together with relations

$$[N, a] = -a, \quad [N, a^\dagger] = a^\dagger. \quad (2)$$

where a, a^\dagger are the annihilation and creation operators, N is the number operator and q is the deformation parameter, which is a c-number. This reduces to the standard Fermi algebra in the limit $q \rightarrow 1$. To proceed further, let us introduce the operator $a^\dagger a = \hat{N}$, with the concomitant action on Fock states, $\hat{N}|n\rangle = \beta_n|n\rangle$, where the eigenvalue depends on n . The relation $\hat{N}a^\dagger + q^{-1}a^\dagger\hat{N} = a^\dagger q^{-N}$ follows from the algebra, Eq.(1). We may set $a|n\rangle = C_n|n-1\rangle$; $a^\dagger|n\rangle = C'_n|n+1\rangle$, where the constants C_n, C'_n can be determined. As a consequence we immediately obtain

$$\beta_{n+1} = q^{-n} - q^{-1}\beta_n. \quad (3)$$

This recurrence relation can be solved in order to determine β_n . We may choose $\beta_0 = 0$, thus defining the ground state as vacuum, i.e., $a^\dagger a|0\rangle = \beta_0|0\rangle = 0$. We accordingly obtain the solution

$$\beta_n = 0, 1, 0, q^{-2}, 0, q^{-4}, \dots = \frac{1 - (-1)^n}{2} q^{-n+1}, \quad (4)$$

which reduces to $\beta_n = 0, q^{-n+1}$ respectively when n is an even or odd number. The action of the creation and annihilation operators on the Fock states yields the results

$$a^\dagger|0\rangle = \sqrt{\beta_1}|1\rangle = |1\rangle; \quad a^\dagger a^\dagger|0\rangle = \sqrt{\beta_1}\sqrt{\beta_2}|2\rangle = 0, \quad (5)$$

the sequence of states thus terminates and consequently the Fock states are $|0\rangle, |1\rangle$ only. In other words, the Fock states in general are built from

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{\beta_n!}} |0\rangle, \quad (6)$$

but restricted to $n = 0, 1$ only. The generalized Fermions thus obey Pauli exclusion principle, just as ordinary Fermions do.

Since q -deformed algebras [3] are in general accompanied by basic numbers [4] (bracket numbers) it is important to stress that there are no basic numbers associated with this deformed algebra. We note that $a^\dagger a \neq [N]_q$, $aa^\dagger \neq [N+1]_q$. Instead, our analysis reveals the operator relations

$$a^\dagger a = \hat{N} = \frac{1 - (-1)^N}{2} q^{-N+1}, \quad aa^\dagger = q^{-N} - q^{-1} \hat{N}. \quad (7)$$

As this algebra is not related to basic numbers, this formulation of q -fermions does not require the use of JD and accordingly we would employ the ordinary derivatives of thermodynamics rather than that of q -calculus. We shall now proceed to investigate the thermostatics of these generalized q -Fermions.

III. THERMOSTATICS OF Q -FERMIONS

From the definition of the expectation value

$$\hat{n} = \frac{1}{\mathcal{Z}} \text{Tr}(e^{-\beta H} \hat{N}) = \frac{1}{\mathcal{Z}} \text{Tr}(e^{-\beta H} a^\dagger a), \quad (8)$$

and from the form of the Hamiltonian $H = \sum_i N_i (E_i - \mu)$, we can determine the distribution function. Using the cyclic property of the trace and the relations $af(N) = f(N+1)$, valid for any polynomial function, we obtain the result

$$\hat{n}_i = \frac{q^{-n_i}}{e^{\beta(E_i - \mu)} + q^{-1}}. \quad (9)$$

Employing the result in Eq.(7), this may be rewritten as

$$\frac{1}{2} (1 - (-1)^n) = \frac{q^{-1}}{e^{\beta(E - \mu)} + q^{-1}}. \quad (10)$$

We may further re-express this result to obtain the distribution function in the form

$$n_i = \frac{2}{\pi} \arcsin \left(\sqrt{\frac{q^{-1}}{e^{\beta(E_i - \mu)} + q^{-1}}} \right), \quad (11)$$

which can then be re-expressed in the form of a power series,

$$n = \frac{1}{\sqrt{g}} + \frac{7\sqrt{g}}{6} + \frac{149g^{3/2}}{120} + \frac{2161g^{5/2}}{1680} + \dots, \quad (12)$$

where $g = q^{-1}/(e^\eta + q^{-1})$. This form may be used to determine all the thermodynamic functions for the q -fermions. However, it is expedient to resort to an approach based on simplicity which exists due to the exclusion principle.

Recalling that the Fock states reduce to $n = 0, 1$ only, we observe that $\sin^2 n\pi/2 = 0, 1$ which can therefore be replaced by n without losing generality. Consequently the distribution function reduces to the simple form

$$n_i = \frac{q^{-1}}{e^{\beta(E_i - \mu)} + q^{-1}}. \quad (13)$$

We may accordingly employ this distribution function following standard procedure [5] to investigate the thermostatics of q -Fermions, noting that we must employ ordinary derivatives and not q -calculus with Jackson Derivatives. The logarithm of the partition function is

$$\ln \mathcal{Z} = \sum_i \ln(1 + q^{-1} z e^{-\beta E_i}), \quad (14)$$

which reproduces the form in Eq.(13), namely

$$n_i = z \frac{\partial}{\partial z} = \frac{q^{-1}}{e^{\beta(E_i - \mu)} + q^{-1}}. \quad (15)$$

Replacing the sum over states by an integration and introducing the thermal wavelength, $\lambda = h/\sqrt{2\pi m k T}$ in the standard manner [5], we determine the expression for the thermodynamic potential

$$\Omega = -\frac{1}{\beta} \ln \mathcal{Z} = -\frac{1}{\beta \lambda^3} \ln(1 + q^{-1} z) - \frac{1}{\beta \lambda^3} f_{5/2}(q, z), \quad (16)$$

where the function f_n defined by

$$f_n(q^{-1} z) = \sum_{r=1}^{\infty} (-1)^{r+1} \frac{(q^{-1} z)^r}{r^n}, \quad (17)$$

is the generalized Riemann Zeta function for q -fermions. The first term in Eq.(16) signifies that we have isolated the zero momentum state in the standard manner. All of these thermodynamic functions reduce to the standard Fermion case in the limit when $q \rightarrow 1$.

The pressure is determined in the thermodynamic limit:

$$P = \lim_{V \rightarrow \infty, N \rightarrow \infty} \left(-\frac{\Omega}{V} \right) = \frac{1}{\beta \lambda^3} f_{5/2}(q^{-1}z), \quad (18)$$

which agrees with the familiar expression in the Fermi limit. The mean density in the thermodynamic limit is given by

$$\frac{n}{V} = \frac{1}{\lambda^3} f_{3/2}(q^{-1}z). \quad (19)$$

To determine how these thermodynamic quantities compare with the corresponding ones for ordinary fermions, we can employ the familiar graphs for the functions $f_{3/2}, f_{5/2}$. Thus the pressure of the q -fermions is greater than that of ordinary fermions at the same temperature and for the same fugacity. Some of the conclusions agree with earlier work [2] but there are significant differences. It is important to stress that the algebra in Eq.(1) for q -Fermions is the same as in ref. [2] but here we have no basic numbers, and no q -calculus with JD.

We shall now examine the virial expansion. In the standard notation, we obtain the result

$$\frac{Pv}{kT} = 1 + \frac{1}{2^{5/2}} \left(\frac{\lambda^3}{v} \right) + \left(\frac{1}{8} - \frac{2}{3^{5/2}} \right) \left(\frac{\lambda^3}{v} \right)^2 + \dots \quad (20)$$

It is interesting to note that the virial coefficients are independent of q , hence do not show deformation. Indeed it is the same as for ordinary fermions and differs from ref. [2]. Furthermore this situation contrasts with the conclusions of earlier work [1] where the formulation was done in two dimensional space based on an ansatz for the distribution function. The present investigation is valid in ordinary 3+1 dimensional space.

The internal energy is given by

$$U = \frac{3kTV}{2\lambda^3} f_{5/2}(q^{-1}z). \quad (21)$$

The entropy of the q -fermion systems is determined by the expression

$$\frac{S}{Nk} = \frac{5}{2} \frac{f_{5/2}(q^{-1}z)}{f_{3/2}(q^{-1}z)} - \ln z. \quad (22)$$

These results possess the expected Fermi limits. We observe that for $q \neq 1$ the entropy is larger than that of ordinary fermions. We shall now state some further general results for the q -fermions.

In the limit of large energy, distribution function reduces to

$$n_i \longrightarrow q^{-1} e^{-\beta E_i}, \quad (23)$$

which, other than the normalization factor, reduces to the quantum Boltzmann statistics. In the limit when $E = \mu$, the distribution reduces to

$$n_i = \frac{q^{-1}}{1 + q^{-1}} \geq \frac{1}{2}, \quad (24)$$

which takes the value $\frac{1}{2}$ only in the Fermi limit when $q = 1$. In the low temperature limit, when $T \rightarrow 0$, it is clear from Eq.(14) that the distribution function reduces to the standard unmodified step form for all values of q . Hence the effect of the deformation may be interpreted solely as a finite temperature effect. The modification at higher temperatures is similar to the standard Fermions except that the parameter q also plays a role.

The dependence on the parameter q is somewhat subtle for many of the thermodynamic functions and it is worthwhile discussing this. As an illustration, let us examine the dependence of the Fermi-energy in some detail. The number density is given by the distribution function

$$\frac{N}{V} = \frac{1}{\lambda^3} f_{3/2}(q^{-1}z), \quad (25)$$

where, for the sake of simplicity, we have omitted the multiplicity factor. This can be expressed by the series

$$\frac{N}{V} = \frac{4\pi}{3} \left(\frac{2mkT}{h^2} \right)^{3/2} (\ln(q^{-1}z))^{3/2} \left(1 + \frac{\pi^2}{8} (\ln(q^{-1}z))^{-2} + \dots \right), \quad (26)$$

and may be employed to determine the chemical potential μ of q -fermions in terms of the Fermi-energy of standard Fermions,

$$E_F = \frac{3N}{4\pi gV}^{2/3} \frac{h^2}{2m}. \quad (27)$$

In the lowest approximation, we obtain

$$\mu = E_F - kT \ln q^{-1}, \quad (28)$$

which shows that the q -dependence appears only at finite temperatures. The expression beyond the zeroth approximation is given by

$$\mu = -kT \ln q^{-1} + E_F \left(1 - \frac{\pi^2}{12} \left(\frac{kT}{E_F} \right)^2 + \dots \right). \quad (29)$$

Thus the temperature dependence of the chemical potential of q -fermions is different from that of standard Fermions for $q \neq 1$.

IV. PARTHASARATHY-VISWANATHAN ALGEBRA

We shall now examine the algebra $ff^\dagger + qf^\dagger f = q^{-N}$, introduced by Parthasarathy and Viswanathan [6], together with the relations $[N, f] = -f$ $[N, f^\dagger] = f^\dagger$. This algebra has also been discussed by Chaichian et al [7], in order to describe fractional statistics. This algebra reduces to the standard Fermi oscillator algebra in the limit $q \rightarrow 1$. Let the operator $\tilde{N} = f^\dagger f$ act on the Fock states $|n\rangle$ so that $\tilde{N}|n\rangle = \alpha_n \tilde{N}|n\rangle$, where the eigenvalue depends on n . The relation

$$\tilde{N}f^\dagger + qf^\dagger\tilde{N} = q^{-N}f^\dagger \quad (30)$$

follows directly from the algebra. If we take $f|n\rangle = C_n|n-1\rangle$, $f^\dagger|n\rangle = C'_n|n+1\rangle$, where C_n, C'_n are constants, we immediately obtain the relation for any n ,

$$\alpha_{n+1} = q^{-n} - q\alpha_n. \quad (31)$$

Solving this recurrence relation, we accordingly determine α_n to be

$$\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = q^{-1} - q, \dots, \alpha_n = q^{-n+1} - q^{-n+3} + \dots q^{n-3} - q^{n-1}. \quad (32)$$

Summing the geometric series, we immediately recognize this as the basic number,

$$\alpha_n = [n] = \frac{q^{-n} - (-1)^n q^n}{q + q^{-1}}. \quad (33)$$

This basic number is characteristic of the algebra and is true for q -fermions, to be contrasted with a different definition applicable for the q -Bosons. This is the definition introduced by Chaichian et al and the solution of the recurrence relation above indeed explains how $f^\dagger f = [N]$ in terms of the basic number, a result which is a direct consequence of the algebra. We further obtain the results

$$f|n\rangle = \sqrt{\alpha_n}|n-1\rangle, \quad f^\dagger|n\rangle = \sqrt{\alpha_{n+1}}|n+1\rangle, \quad ff^\dagger = [N+1]. \quad (34)$$

Let us first examine the Fermi limit when $q \rightarrow 1$. We find $\lim_{q \rightarrow 1} \alpha_2 = 0$; other α_n may be non-zero in the limit. In the limit, when $q \rightarrow 1$, Fock states are therefore restricted to $|0\rangle, |1\rangle$ only and Pauli exclusion principle is valid only in the limit. For arbitrary values of q , we may construct the Fock states by

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]!}}|0\rangle, \quad (35)$$

where $[n]! = [n] \cdot [n-1] \cdots [1]$. We observe that this is different from the rationale provided in ref. [7] for invoking the exclusion principle in the limit, hence we need not assume the relation $f^2 = (f^\dagger)^2 = 0$. We also note that $[n] = \frac{1}{2}(1 - (-1)^n)$ in the limit $q \rightarrow 1$. As a consequence, the Fock space breaks up into an infinity of 2-dimensional subspaces when $q = 1$, with the Pauli principle valid in each subspace. On the other hand, $|n\rangle$ exists and is non-zero for arbitrary n when $q \neq 1$. Consequently these are generalized fermions with $n = 0, 1, 2, \dots$.

The Hamiltonian of the generalized Fermions may be taken to be $H = \frac{1}{2}\hbar\omega([N] - [N+1])$ with $E = \frac{1}{2}\hbar\omega([n] - [n+1])$, and hence there is no equal spacing rule for arbitrary q . However, exclusion principle prevails and $E \rightarrow -\frac{1}{2}\hbar\omega, +\frac{1}{2}\hbar\omega$ for $n = 0, 1$ in the limit.

As discussed in ref. [7], the Fock space for complex q , is quite tricky. When $q = e^{i\pi/m}$, we find $[2m] = 0$ when $n = 2m = 2 \times \text{odd}$. And when $n = m = 4k$, we find $[m] = 0$. However, this feature is not present when q is real and therefore the formulation with real q has distinct advantages.

The basic number here exhibits skew symmetry i.e., $[n] \longrightarrow \pm[n]$ for $n = \text{odd}$, even and this contrasts with the situation in other algebras. e.g.,

$$[n]_B = \frac{q^n - q^{-n}}{q - q^{-1}} \implies [n]_B(q^{-1}) = [n]_B(q). \quad (36)$$

The thermodynamics of these generalized Fermions would involve q -calculus with JD. However, special care is needed in order to identify q -calculus with JD in this formalism, due to the factor $(-1)^N$ in Eq.(32). In order to establish the JD for the q -Fermion algebra, we proceed to analyze as follows.

First, we recall the JD in the q -boson case:

$$\mathcal{D}f(x) = \frac{1}{x} \frac{q^N - q^{-N}}{q - q^{-1}} f(x) = \frac{f(qx) - f(q^{-1}x)}{x(q - q^{-1})}, \quad (37)$$

which reduces to the ordinary derivative in the limit $q \rightarrow 1$. In order to study the case of q -fermions, arising from the algebra $ff^\dagger + qf^\dagger f = q^{-N}$, we may invoke the holomorphy relation $f \iff \mathcal{D}_x$, $f^\dagger \iff x$ due to which the algebra implies $\mathcal{D}_x x + qx\mathcal{D} = q^{-N}$. It may be useful to recall: $f \iff \partial/\partial x$, $f^\dagger \iff x$, $N = f^\dagger f$, leads to properties [8] such as

$$q^N x = xq^{N+1}; \quad q^N x^r = (qx)^r, \quad (38)$$

etc. From the property $[N]x = x[N+1]$, i.e., $[N]x + qxN = xq^{-N}$, we infer the solution of $\mathcal{D}_x x + qx\mathcal{D} = q^{-N}$ to be

$$\mathcal{D}_x = \frac{1}{x} \frac{q^{-N} - (-1)^N q^N}{q + q^{-1}} \quad (39)$$

as the appropriate JD for q -fermions. If we now employ the properties

$$\begin{aligned} q^N f(x) &= f(qx), \\ q^{-N} f(x) &= f(q^{-1}x), \\ (-q)^N f(x) &= f(-qx), \end{aligned} \quad (40)$$

this can be expressed as a differential operator in the standard manner,

$$\mathcal{D}_x f(x) = \frac{1}{x} \frac{f(q^{-1}x) - f(-qx)}{q + q^{-1}}, \quad (41)$$

valid for q -fermions. This may be contrasted with the JD used in q -boson calculus, where

$$\mathcal{D}_x f(x) = \frac{1}{x} \frac{f(qx) - f(q^{-1}x)}{q - q^{-1}}. \quad (42)$$

For the Fermion case, one can investigate many of the properties satisfied by the JD. In particular the q -Fermion JD reduces to the ordinary derivative when $q \rightarrow 1$: that is, $\lim_{q \rightarrow 1} \mathcal{D}_x f(x) = f'(x)$ which is easily established using L'Hospital rule.

V. SUMMARY AND CONCLUSION

We have investigated the consequences of the q -deformed algebra $a^\dagger + q^{-1}a^\dagger a = q^{-N}$, $0 \leq q \leq 1$ describing generalized Fermions which obey the exclusion principle. In addition to presenting the mathematical formulation, we have considered detailed physical applications of the generalized Fermions and obtained the various thermodynamic functions so that we can see what the physical consequences are. The algebra, together with all the thermodynamic consequences of the system of q -deformed Fermions reduce to those of ordinary Fermions in the limit $q \rightarrow 1$. We have determined the thermodynamic functions such as the partition function, pressure, and the entropy. We have also determined the dependence on q of the chemical potential as a function of temperature. This is an example where the deformation is seen to be a finite temperature effect.

The Fock states are constructed by $|n\rangle = (a^\dagger)^n / \sqrt{\beta_n!} |0\rangle$, where β_n depends on q and $\beta_n = 0, 1$ for $n = 0, 1$. The thermodynamic properties of these Fermions are dependent on the deformation parameter. However, the algebra nevertheless has no basic numbers associated with it and the system is governed by the ordinary calculus of thermodynamics and not the q -calculus in terms of the JD, in contrast to the earlier work cited [2]. The q -Boson algebra corresponds to the basic number $[n] = (q^n - q^{-n})/(q - q^{-1})$ but the q -fermions are not associated with any basic number. This enables us to understand how the thermodynamic properties of the generalized Fermions could be different from systems governed by basic numbers and by JD.

We have also investigated the generalized Fermions, not obeying the exclusion principle, stemming from a q -deformed oscillator algebra. We have established the following basic premises. The Fock states of these generalized Fermions can be built from the action of the creation operators and require the use of Fermion basic numbers which follow directly from the algebra. The q -calculus needed to study the thermostatics of these Fermions must employ a JD which is characteristic of the nature of the generalized Fermions. We have determined the form of this JD which reduces to the ordinary derivative in the Fermi limit, $q \rightarrow 1$. The basic numbers as well as the JD occurring in the formulation of these generalized Fermions are quite distinct from corresponding results for the q -deformed Bosons known in the literature.

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